HOW TO CALCULATE INTEREST

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Contents

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0. Magic Formulas

Fixed-Rate Compound Interest

assuming the borrower has not paid back any of the loan.

$$A(t) = A_0\left(1 + \frac{R}{n}\right)^{nt}$$

A_0	principal amount borrowed
R	Annual Percentage Rate
n	number of compounding periods per year
$A(t)$	amount owed after t years

APR vs. APY comparison

Y = Annual Percentage Yield

$$Y = \left(1 + \frac{R}{n}\right)^{n} - 1 \qquad R = n\left(\sqrt[n]{Y+1} - 1\right)$$

Example: $10,000 loan at 6.00% APR

$$A_0 = \$10,000 \qquad r = .06 \qquad n = 12$$

$$A(1) = 10,000 \times \left(1 + \frac{.06}{12}\right)^{12} \approx 10,616.78$$

Interest owed after 12 months: $616.78 APY ≈ 6.1678%

Monthly Payments (Approximate)

Continuous payments totaling M per month for T years.

WARNING: May underestimate actual monthly payments!

$$M = A_0\frac{R}{12}\left(\frac{1}{1 - e^{-RT}}\right) \qquad T = \frac{-1}{R}\ln\left(1 - \frac{A_0 R}{12M}\right)$$

Fixed-Rate Savings Account (Approximate)

Amount in account after t years with monthly compounding and deposits of size D every month. (Does not include inflation or taxes!)

$$A(t) = \left(A_0 + \frac{12D}{R}\right)\left(1 + \frac{R}{12}\right)^{12t} - \frac{12D}{R}$$

Mortgage Payments (Exact)

Monthly compounding, one payment M per month for T years, no fees.

$$M = A_0 \frac{R}{12}\left(\frac{1}{1 - \left(1 + \frac{R}{12}\right)^{-12T}}\right)$$

$$T = \frac{-1}{12\log(1 + \frac{R}{12})} \log\left(1 - \frac{A_0 R}{12M}\right)$$

Stafford Loans (Exact)

During grace periods, the balance on *subsidized* loans does not change. The balance on *unsubsidized* loans increases. After m months, it is

$$A(m) = A_0\left(1 + \frac{R}{12}m\right)$$

Otherwise, treat a Stafford loan as a monthly-compounded mortgage.

Credit Card Payments (Approximate)

Daily compounding, one payment M per month for T years, no fees. Total charges of C per month with no late payments or defaults.

$$M = A_0\left(\frac{x-1}{1 - x^{-12T}}\right) + C$$

$$T = \frac{-1}{12\log(x)} \log\left(\frac{A_0(x-1)}{C - M}\right)$$

In both of these formulas, x is defined in terms of R:

$$x \equiv (1 + \frac{R}{365})^{31} \approx 1 + .088(R)$$

Example Spreadsheets

Each row represents one month. For all cells in a column, use these formulas.

Mortgage (Monthly Compounding)

$$\begin{aligned}
\text{Balance[m]} &= \text{Balance[m-1]} + \text{Interest[m-1]} - \text{Payment[m-1]} \\
\text{Interest[m]} &= \text{Balance[m]} * \tfrac{1}{12}\text{Rate[m]} \\
\text{Rate[m]} &= \text{APR for that month} \\
\text{Payment[m]} &= \text{whatever amount was paid that month}
\end{aligned}$$

Variable-Rate Savings Account (Monthly Compounding)

$$\begin{aligned}
\text{Balance[m]} &= \text{Balance[m-1]} + \text{Interest[m-1]} + \text{Deposit[m-1]} \\
\text{Interest[m]} &= \text{Balance[m]} * \tfrac{1}{12}\text{Rate[m]} \\
\text{Rate[m]} &= \text{APR for that month (Use APR vs. APY formula.)} \\
\text{Deposit[m]} &= \text{whatever amount was deposited that month}
\end{aligned}$$

Unsubsidized Stafford Loans During Grace Period

$$\begin{aligned}
\text{Balance[m]} &= \text{total amount borrowed to date} \\
\text{Interest[m]} &= \text{Balance[m]} * \tfrac{1}{12}\text{APR}
\end{aligned}$$

When grace period ends, find SUM() of Interest column. Add this to Balance and begin a new spreadsheet, treating all loans as one fixed-rate mortgage.

Credit Card (Daily Compounding, Approximate)

$$\begin{aligned}
\text{Balance[m]} &= \text{Balance[m-1]} + \text{Charges[m-1]} + \text{Interest[m-1] - Payment[m-1]} \\
\text{Interest[m]} &= (\text{Balance[m]} + \text{Charges[m]}) * (1\text{+DailyRate[m]}) \wedge (30.5) \\
\text{DailyRate[m]} &= \text{Daily Periodic Rate for that month (usually = APR / 365.)} \\
\text{Charges[m]} &= \text{Total charges during month } m \\
\text{Payment[m]} &= \text{Total payments made during month } m
\end{aligned}$$

Approximations: 30.5 days per month, all charges made on first day of month, payment accepted only at end of month, card is never in grace period or default, no fees. This approximation should slightly overestimate actual credit balances.

Disclaimer: This document is not intended as financial advice. It is intended as educational material for people who wish to become more proficient with financial calculations and/or to better understand the mathematical basis of modern finance. The author has degrees in mathematics and physics but makes no claims of any professional financial or legal training or certification. Laws and financial conventions can change rapidly and the examples herein may become outdated.

Caveat lector!

1. Introduction

MACHIAVELLI: I fear that you have some prejudice against loans. They are precious for more than one reason: they attach families to the government; they are excellent investments for private citizens; and modern economists today formally recognize that, far from impoverishing the States, public debts enrich them. Would you like to permit me to explain how to you?

-The Dialogue in Hell Between Machiavelli and Montesquieu, Maurice Joly, 1864 as translated from the French by anonymous authors at www.notbored.org, 2008.

Loans, bonds, credit, and mortgages can be complicated, but the basic idea is simple: a lender gives money to a borrower, then the borrower pays the lender more money in the future. All good loans have three things in common:

(1) The borrower needs money sooner rather than later.
(2) The lender can afford to risk lending money to the borrower.
(3) The borrower has a realistic plan for paying the lender back.

Bad loans break at least one of these rules.

For example, Alice makes delicious cake and could earn money by opening her own bakery. Sadly, Alice must first spend years saving money because she cannot afford the start-up costs. If she had money now, she could start the business now and profit more. This is what economists mean by the **time value of money**.

Bob can afford to open a bakery, but he is an inept baker. He offers to lend Alice money and she agrees to use her future profits to pay back more than she has borrowed. The amount Alice borrows is the **principal** of the loan, and whatever extra money she pays Bob is **interest**. Interest is Bob's compensation for exposing his money to **risk**: if for some reason Alice does not pay him back, he will lose money. To reduce his risk, Bob asks for the bakery as **collateral**: if Alice does not pay on time, Bob will own the bakery. If Bob asks for too much interest or charges too much in fees, another investor might offer Alice a better deal and Bob will get nothing. If Bob doesn't charge Alice enough, he faces **opportunity costs**: he could profit more by investing elsewhere.

2. Simple Interest: Bonds and Flat-Rate Loans

$$A(t) = A_0(1 + rt)$$

A_0	amount borrowed
r	interest rate per time period
$A(t)$	amount owed after t time periods

Bonds are possibly the simplest modern example of loans with interest. Since the author lives in the United States, we'll start with U.S. Treasury bonds, notes, and bills. These are loans from the bond-buyer to the U.S. federal government.

U.S. Treasury bills, also called T-bills, are simple **one-term** loans. For example, a 3-month \$1,000 T-bill might cost a buyer \$980. After 3 months, the treasury pays **face value** of \$1,000 to the bond owner. T-bills are used when the government has a short-term need for cash. By selling this bond, the government says it is willing to pay someone \$20 to have \$980 now rather than three months in the future.

T-bonds and T-notes are called **coupon bonds**. These are longer-term loans in which the government pays the owner a fixed "coupon" payment every 6 months until a fixed time called the **maturity date**. For example, a \$10,000 bond with a 5% coupon would pay $.05 \times 10,000 = \$500$ per 6-month period. Once the bond "matures," the government pays back the **par value** (a.k.a. face value) of \$10,000.

If a bond with a coupon rate of r is held for t time periods and then redeemed for par value A_0, the total amount paid from the bond issuer to the buyer is

$$A(t) = A_0(1 + rt)$$

For a \$10,000 5-year T-note with a 2% coupon, $r = 0.02, A_0 = 10,000$, and $t = 10$ is the number of 6-month periods in 5 years. The total amount paid by the government is $10,000(1 + .02 \times 10) = 12,000$ dollars, so buying this bond would yield a profit of \$2,000. (The practical value of this profit depends on taxes and inflation rates.) Bond owners can legally sell their bonds to other investors, so the market value of any given bond changes over time depending on investors' opinions about interest rates, inflation, and other information. Whoever owns the bond when it matures receives the final face-value payment, so bond prices tend to converge towards face value.

Buying a T-bond or T-note is the same as making a **balloon loan** with a fixed interest rate to the U.S. Treasury. "Balloon" simply means that the face value of the loan is paid off in one big lump by a specified date. Because of their all-or-nothing repayment scheme, balloon loans to private citizens can be very risky for the lender, especially if the borrower is unable to pay at the specified maturity date. U.S. bonds, however, are considered very low-risk investments. The federal government has a long history of paying its debts on time, not to mention the ability legally print money. Of course, printing money leads to inflation, which devalues bonds - so what really decides bond prices is investors' confidence in the financial security of the government itself.

Flat-rate loans are a related type of credit. A lender lends some principal amount and charges a fixed interest rate per time period. The difference is, the borrower is expected to pay back some portion of the loan each period. Flat-rate and balloon loans differ in their **amortization schedules**:[1] flat-rate loans require a series of fixed payments, while balloon loans require one lump sum. For example, Bob offers Alice the following flat-rate 12% loan with 10 annual payments:

(1) Alice borrows $100,000 from Bob and opens a bakery.
(2) Alice pays Bob $10,000 plus 0.12 × $100,000 each year.
(3) After 10 payments, Alice's debt to Bob is paid and the bakery is hers.
(4) Alice must not miss a payment, or Bob gets the bakery.

The total cost to Alice after 10 years is $\$100,000 \times (1 + 0.12 \times 10) = \$220,000$.

Flat-rate loans are simple but inflexible. If Alice has a bad month and misses a payment, Bob repossesses the bakery and the loan fails. Contrariwise, if Alice's shop performs better than expected, Alice has no incentive to repay part of her loan early. Alice then pays unnecessary interest and Bob suffers opportunity costs. (Also note that Alice's so-called "12% loan" will cost her $220,000 = 220% of the principal! For this reason, flat-rate loans are considered deceptive and illegal in many nations.)

Alice's needs (low minimum payments, less interest) and Bob's needs (less risk, faster repayment) might be better met by a modern mortgage with compound interest.

3. Mortgages

Consider the following example of a commercial mortgage: Bob offers Alice a loan with monthly-compounded 1% interest[2] using her bakery as collateral. (A homeowner's mortgage or automobile financing is often a similar deal with one's house or car as collateral.) Each month, Alice must pay a small minimum amount. She can also choose to pay more, in which case she pays less total interest and Bob gets some of his money back early. The simplest minimum payment is zero, so we'll consider that case first.

(1) Alice borrows $100,000 from Bob. This is her principal balance.
(2) Each month's interest is 0.01×(Alice's previous balance).
(3) Alice pays whatever amount she wants to pay during that month.
(4) At the end of the month, Bob recalculates the balance using this formula:

$$\text{New Balance} = \text{Previous Balance} + \text{Interest} - \text{Last Payment}$$

[1] An amortization schedule is a plan for repaying a loan over a fixed time period. This and the word "mortgage" come from the Old French *morte gage* meaning "dead [i.e. expired] obligation."

[2] This is equivalent to a 12% APR. See the section "APR vs. APY" for details.

(5) Repeat steps 2,3, and 4 until Alice's balance is zero.

The table below shows one possible amortization schedule for Alice's loan. (Tables like this can be made with OpenOffice, Microsoft Excel, Apple iWork, or other spreadsheet software.) For month number m, the numbers in each column are:

$$\begin{aligned}\text{Balance[m]} &= \text{Balance[m-1]} + \text{Interest[m-1]} - \text{Payment[m-1]}\\ \text{Interest[m]} &= \text{Balance[m]} * 0.01\\ \text{Payment[m]} &= \text{whatever amount Alice paid that month}\end{aligned}$$

Example spreadsheet: 12% APR mortgage, no minimum payment.

Month	Balance	Interest	Payment
0	100,000.00	-	-
1	100,000.00	1,000.00	0
2	101,000.00	1,010.00	0
3	102,010.00	1,020.10	1,020.10
4	102,010.00	1,020.10	1,020.10
5	102,010.00	1,020.10	1,020.10
6	102,010.00	1,020.10	5,000.00
7	98,030.10	980.30	5,000.00
8	94,010.40	940.10	5,000.00
9	89,950.51	899.51	20,000.00
10	70,850.01	708.50	20,000.00
11	51,558.51	515.59	20,000.00
12	32,074.10	320.74	32,394.74
13	0	0	0

Total paid by Alice to Bob: $110,455.04

Month 0 here means "0 months have passed," i.e. the day Alice borrowed the principal balance of $100,000. Some loans charge an **origination fee** on the first day. For example, Bob might keep 2% of the original balance ($2,000) but charge interest on the entire $100,000 anyway. In our example, however, Bob charges no such fee.

According to the table, Alice paid Bob nothing during the first two months. After 1 month, she owed 1% of $100,000 = $1,000 in interest. Bob **capitalized** this interest: he added it to her balance. After 2 months, she owed another $1,010. Her next interest payment is $1,020.10. Notice that Alice is now paying interest on her previous months' interest - in other words, her interest is being **compounded**.

If Alice always pays less than the new interest charged each month, she will never pay off the loan. This is called **negative amortization**. Alice isn't killing her debt -

she's making it bigger![3] If Alice continues to pay nothing, her balance after 6 months is:

$$A_0 \times (1.01) \times (1.01) \times (1.01) \times (1.01) \times (1.01) \times (1.01) = A_0(1.01)^6$$

Alice would owe Bob $\$100,000 \times (1.01)^6 = \$106,152.02$. After a year, the amount is $\$100,00 \times (1.01)^{12} = \$112,682.50$ and after 10 years, her debt would be $\$100,000 \times (1.01)^{120} = 330,038.69$. Negative amortization is clearly not a sustainable strategy for paying back a loan, though it can sometimes be reasonable for a limited period of time.

In our example, Alice pays exactly $1,020.10 for the next three months. These **interest-only** payments prevent her debt from growing or shrinking. At this rate, Alice will still be paying interest to Bob forever. Interest-only payments are like treading water; they keep Alice from sinking, but she's not getting any closer to shore.

After those three months, Alice's bakery becomes profitable and she decides to pay Bob $5,000 per month. Notice that her balance drops steadily, but slowly. Almost 20% of each payment goes towards paying interest instead of paying off the principal. By Month 9, Alice starts to pay Bob $20,000 per month. More than 95% of each of these payments goes towards paying back the principal. Finally, after one year, Alice has enough cash on hand to end the debt permanently. She pays her balance plus her last interest payment and is out of debt. Bob's profit is $10,455.04.

Real mortgages are often **amortizing** (a.k.a. **declining-balance**) loans which require monthly payments to always be larger than monthly interest. The monthly interest charges thus decline over time as the balance decreases. By contrast, interest payments do *not* decline along with the balance in a flat-rate loan. In this example, Alice amortized her loan quickly once her business became profitable. Consequently, her total interest is *much* lower than it would be from a flat-rate 12% loan: $10,455.04 versus $120,000.00.

This example mortgage is a **fixed-rate** loan, meaning the interest rate of 12% APR never changes. (The precise meaning of "APR" is described the subsection "APR vs. APY." For now, just think of it as an annual interest rate.) Other loans may use **variable interest rates** which may change during the loan according to some specific rules. Variable rates are often indexed to major banks' and governments' interest rates so the rates will rise and fall along with the general cost of borrowing money nationwide.

For example, a loan might offer an APR of "LIBOR plus 4.99%," which means the APR is always 4.99% plus the London InterBank Offered Rate. The U.S. Federal Reserve (a.k.a. "the Fed") also sets its own interest rate. North American banks often use the U.S. **prime rate**, a weighted average of rates offered by large banks to their lowest-risk customers. The prime rate is usually strongly influenced by the Fed rate, which gives the Fed some control over national interest rates.

[3] Even worse, the rate at which her debt grows steadily increases. This is covered in detail in the appendix "Mathematics of Exponential Growth."

To calculate interest on a variable-rate loan using a spreadsheet, we'll need an extra column containing the interest rate. Consider a loan with an APR of "prime + 5.99%," compounded monthly, with a principal balance of $100,000:

$$\begin{aligned}
\text{Balance[m]} &= \text{Balance[m-1]} + \text{Interest[m-1]} - \text{Payment[m-1]} \\
\text{Interest[m]} &= \text{Balance[m]} * \text{Rate[m]} \ / \ 12 \\
\text{Payment[m]} &= \text{whatever amount Alice paid that month} \\
\text{Rate[m]} &= .0599 + \text{whatever the prime rate is for that month}
\end{aligned}$$

Alice uses the same amortization schedule as before: no payments for 2 months, interest-only for 3 months, $5K for 3 months, $20K for 3 months, then one big payment for the remaining balance. (I made up some numbers for the prime rate; it was 2.50% for 4 months, then 3.00% for 6 months, then 3.25%.)

Month	Balance	Interest	Payment	Rate
0	100,000.00	-	-	.0849
1	100,000.00	707.50	0	.0849
2	101,707.50	712.51	0	.0849
3	101,420.01	717.55	717.55	.0849
4	101,420.00	717.55	717.55	.0849
5	101,420.00	759.80	759.80	.0899
6	101,420.00	759.80	5,000.00	.0899
7	97,179.81	728.04	5,000.00	.0899
8	92,907.85	696.03	5,000.00	.0899
9	88,603.88	663.79	20,000.00	.0899
10	69,267.67	518.93	20,000.00	.0899
11	49,786.60	383.36	20,000.00	.0924
12	30,169.96	232.31	30,402.27	.0924
13	0	0	0	-

Total paid by Alice to Bob: $107,599.45

Lenders should specify how they define "prime rate." One common definition is "whatever number the Wall Street Journal publishes under the name 'prime rate' on the first day of the month." Of course, we don't know what the prime rate will be ahead of time - that's part of the risk of variable-rate loans.

4. Savings Accounts

Checking accounts allow customers to deposit money and withdraw cash or write checks, often at no charge to customers.[4] Savings accounts pay interest on deposits. A savings deposit is effectively a loan from the customer to the bank - but unlike a mortgage, the lender decides the size of the balance each month.

[4]Fees often apply when customers withdraw more money than is in the account, write bad checks, use other banks' ATM machines, convert to foreign currencies, and so forth.

Savings account interest rates are usually listed in terms of APY. If an amount A_0 is deposited in a savings account and left there for one year at a fixed rate, it will increase in value by an amount $A_0 \times$APY. In practice, savings accounts usually offer monthly-compounded variable-rate interest, which complicates the formula somewhat. (Precise definitions and conversion formulas are shown in the section "APR vs. APY.")

In the U.S., the government-owned **Federal Deposit Insurance Corporation** insures savings accounts for up to $250,000 in case a bank is unable to pay when a customer requests a withdrawal.[5] Savings accounts are thus extremely low-risk investments and consequently offer low interest rates.

Savings accounts offer excellent **liquidity**, meaning they can be converted into cash ("liquidated") much more quickly than long-term investments like real estate or Alice's bakery business. However, savings accounts are generally designed (and often legally required[6]) to limit withdrawals. Customers are expected to use checking accounts for paying bills and everyday expenses, and savings accounts may include rules such as minimum balance levels or a maximum number of withdrawals per month.

Similarly, **certificates of deposit** are FDIC-insured bank deposits that cannot be withdrawn for a fixed period of time unless the customer pays a penalty fee. Customers can often choose whether CD interest is paid monthly like a coupon bond or added to the principal and compounded. CDs are more predictable for the bank but less accessible to the customer and generally pay higher interest rates than savings accounts. **Online savings accounts** also offer higher interest rates but require customers to access their account electronically, which lowers the bank's costs for real estate and labor. Another savings-like account is a **money market deposit account**, which is also insured by the FDIC and includes limits on withdrawals. (These should not be confused with money market *funds*, which are not FDIC-insured and may lose value.)

Including taxes and inflation, the effective long-term returns from savings accounts, CDs, and money market deposits are often underwhelming. Historically, savings accounts have existed not to make huge profits but primarily to protect personal savings from inflation and/or theft. The idea is simple: a steady trickle of a few percent per year is not a huge profit, but it's safer than hiding bundles of cash under a mattress.

5. APR vs. APY

Loans in the U.S. are often required by law[7] to disclose an **Annual Percentage Rate**, while savings accounts and other investments usually advertise an **Annual Percentage Yield** or **Effective Annual Rate.**[8] Here are the definitions:

[5] $250,000 as of 2010. This number is scheduled to be reduced to $100,000 by 2014.

[6] See the Glass-Steagall Act of 1933, especially Regulations Q and D and later modifications.

[7] For details, see the Truth In Lending Act of 1968.

[8] EAR is usually defined exactly the same as APY.

APR: If a loan is compounded n times per year at interest rate r, then

$$APR = nr$$

For a loan with monthly compounding, APR = 12 × the monthly interest rate. For daily compounding, APR = 365.25 × the daily interest rate.[9]

APY: If a borrower makes no payments on a loan with principal A_0, then

$$A_1 = A_0(1 + APY)$$

is the size of the debt after exactly one year. For example, if you deposit \$100 in a savings account with a fixed APY of 2% and withdraw nothing for one year, you will have \$102 in your account. Your profit is A_0×APY = \$100 × 0.02 = \$2.

For short periods of time and low interest rates, APR and APY are approximately equal. To see the difference, consider a very simple loan: a lender offers A_0 to a borrower for exactly 1 year, and neither makes any payments to the other. The amount owed to the lender after one year is $A_1 = A_0$(1+APY). In terms of APR, that same amount is:

$$A_1 = A_0\,(1 + r)^n = A_0\Big(1 + \frac{APR}{n}\Big)^n$$

From the definitions of APR and APY, we can convert an APR into an APY.

$$A_0(1 + APY) = A_0\Big(1 + \frac{APR}{n}\Big)^n \quad \Rightarrow \quad APY = \Big(1 + \frac{APR}{n}\Big)^n - 1$$

Using some algebra, we can convert from APY to APR as well:

$$\sqrt[n]{1 + APY} = 1 + \frac{APR}{n} \quad \Rightarrow \quad APR = n\big(\sqrt[n]{1 + APY} - 1\big)$$

The difference between APR and APY depends on how large the interest rate is and how often it is compounded. Figure 1 shows APY as a function of APR for a one-year loan with no payments using daily, monthly, and annual compounding. For annual compounding, APY = APR. For more rapid compounding, APY > APR. As interest rate rises and compounding frequency increases, the two drift apart. Compounding daily rather than monthly results in a subtle, but noticeable, difference.

Compounding more often means more interest is charged over the same period of time. Perhaps surprisingly, compounding extremely rapidly produces only a small difference in total interest. For realistic interest rates, a lender who compounds interest every second of every day will make only slightly more money than one who compounds daily. (The difference would not be visible in Figure 1.) As compounding frequency increases, the conversion between APY and APR approaches a limit:[10]

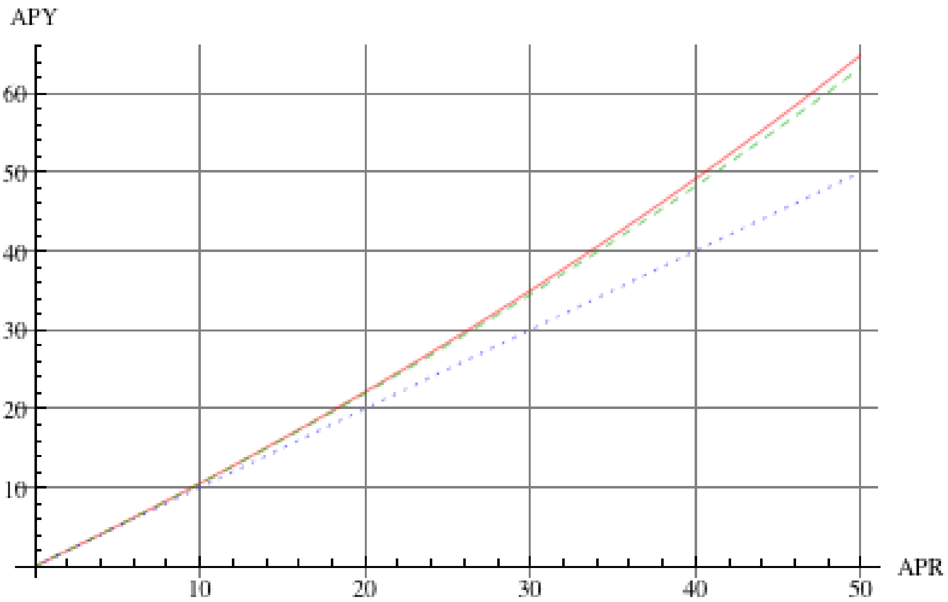
$$\lim_{n\to\infty} APY = e^{APR} - 1 \qquad\qquad \lim_{n\to\infty} APR = \ln(1 + APY)$$

Here $e \approx 2.71828$ is the base of the natural logarithm function, ln(). These equations for **continous compounding** can be used as a quick estimate for APY/APR conversion.

[9] There are various different ways to take leap years into account; this is just one of them.

[10] Details about this formula are in the appendix "Mathematics of Exponential Growth."

Figure 1. 1-Year APY vs. APR comparison for $0 < \text{APR} < 50\%$.



Top to bottom: daily, monthly, and annual compounding.

Continuous compounding always overestimates APY for a given APR (and underestimates APR for a given APY), but in practice, it's often a good approximation.

6. Student Loans

Federal student loans in the U.S. use the federal government as a **guarantor**: the government guarantees lenders will be repaid even if students are unable to pay off their loans.[11] Offering student loans to students, regardless of their credit history or ability to pay, is therefore a low-risk investment for lenders.

Another feature of federal student loans is a relatively long **grace period**, during which time borrowers are not required to make payments. Student loans **accrue** interest during this time but do not compound it.[12] For most student loans, grace periods occur any time a student is enrolled at least half-time at an accredited college or university.[13] For Perkins and subsidized Stafford loans, the government makes interest-only payments at no charge to the student during grace periods.[14] However, once a student loan exits

[11]Students who default don't get a free ride. The government can garnish money from students' paychecks, reduce their tax refunds, and even imprison students who fail to pay.

[12]The interest charged each month is ($\frac{1}{12}$APR $\times$ principal) instead of ($\frac{1}{12}$APR $\times$ balance).

[13]Stafford and Perkins grace periods end 6 and 9 months after students graduate or drop out.

[14]These loans are need-based. Unsubsidized Stafford loans are not need-based.

a grace period, any unpaid interest is **capitalized**: it is added to the balance and compounding begins. Students are usually allowed but not required to lower the total cost of each loan by making interest-only payments during grace periods.

Stafford and Perkins loans limit borrowing amounts depending on the number of years a student has been enrolled. For those wishing to borrow more than these limits, federally-guaranteed PLUS loans are available. These may require parents to **co-sign**, which means parents share responsibility for payment and undergo a credit check. PLUS loans generally demand higher rates than Stafford and Perkins loans. Many lenders and some universities offer private loans in addition to federally-guaranteed loans, but these usually demand higher interest rates than PLUS loans.

Stafford, Perkins, and PLUS loans typically calculate minimum payments such that the loan must be completely repaid after 10 years. Federal and private **consolidation** loans also available for longer repayment periods of 10 to 30 years. Loan consolidation replaces all existing student loans with one large loan, typically featuring a lower minimum monthly payment over a longer period of time for a higher total cost.

Here's an example calculation: an undergraduate student borrows the maximum allowed each semester (2 semesters per year) in unsubsidized 10-year Stafford loans with a fixed 6.8% APR. The student makes no payments while in school, graduates after 4 years, and begins making constant minimum monthly payments after the 6-month grace period ends. An origination fee of 3% is taken out of these loans when they are **disbursed**, i.e. given to the student. (Thus the student receives \$26,190 but pays interest on a \$27,000 principal.) The maximum borrowing amounts are:

Year	1st Semester	2nd Semester
1	2,750	2,750
2	3,250	3,250
3	3,750	3,750
4	3,750	3,750

Principal: \$27,000

Using a spreadsheet to calculate interest after 54 months gives a principal of \$27,000 plus \$4,887.50 in interest for a total balance of \$31,887.50:

Balance[m] = manually input from previous table, do *not* add interest
Interest[m] = Balance[m] ∗ .068 / 12

Now we'll calculate a monthly payment such that this loan can be paid off in 66 months = 10 years minus 48 months of school minus 6 months of grace period. This can be done using the Magic Formula for mortgage monthly payments:

$$M = A_0 \frac{R}{12} \left(\frac{1}{1 - (1 + \frac{R}{12})^{-12T}} \right) = 31,887.50 \left(\frac{.068}{12} \right) \left(\frac{1}{1 - (1 + \frac{.068}{12})^{-66}} \right) \approx \$580.46$$

Assuming the student never misses a payment, the total cost over 10 years is about

$\$580.46 \times 66 \approx \$38{,}300$. The lender's profit (neglecting inflation) is about \$12,100, or 46% of the initial \$26,190 loan. Consolidating the loan for 30 years instead results in a monthly payment of \$219.68 over 306 months for a total cost of about \$67,200.

7. Credit Cards

Credit cards allow cardholders to avoid temporary shortages of cash during emergencies, vacations, holidays, and unexpected circumstances. High interest rates are the price of this convenience. When used incautiously, credit cards can drown unwary cardholders in perpetual exponentially-increasing debt. The total credit card debt of all Americans was almost \$1 trillion at the end of 2008.[15]

A **charge card** allows customers to make cashless purchases and delay payment, usually until the end of each month. Charge cards reduce consumers' need to use checks or carry large amounts of cash. Card issuers profit by charging merchants some percentage of each sale and/or requiring fees from customers. Charge cards typically assess large penalty fees if a customer's charges are not paid in full each month.

A **credit card** functions like a charge card, but customers are not required to pay for their purchases by any specific deadline. Cardholders who do not pay their balances in full must pay interest on any unpaid charges. Many credit cards use an **average daily balance** method: each charge made is multiplied by the number of days remaining in that month, then these numbers are added together and divided by the number of days in that month. Card issuers then calculate interest as if the customer charged exactly the average daily balance every day during that month.[16]

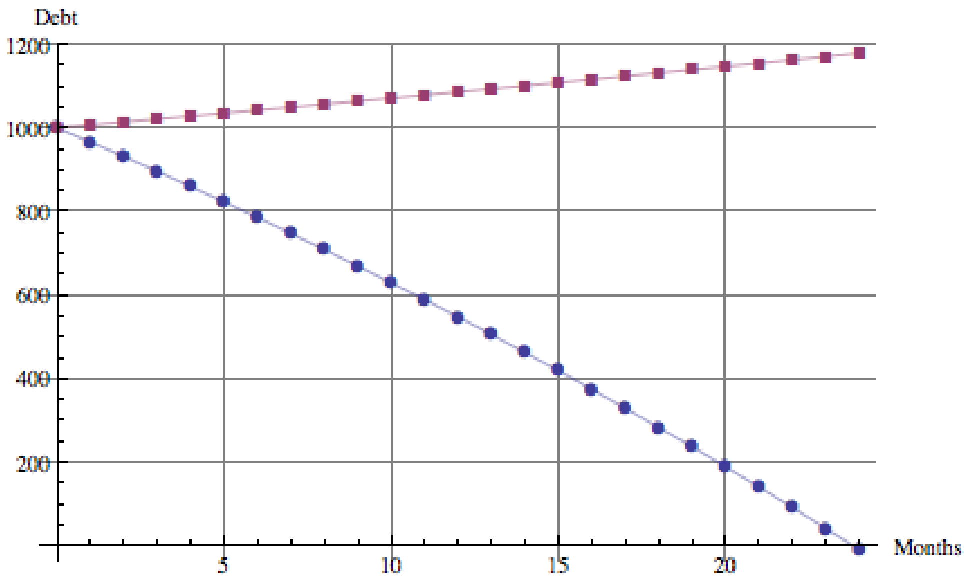
Grace periods for cards typically last about a month. If a customer always pays his or her charges in full during the grace period, no interest is required and the card functions as a charge card. If any unpaid charges remain after the grace period ends, then interest charges begin. Customers who pay less than their full balance each month are said to **revolve** their debt or "carry a balance." Cardholders may revolve debt if they continue to make on-time minimum payments and the balance does not exceed a **credit limit** determined by the card issuer. Usually, a new grace period begins once the customer pays off the entire balance. As an incentive to switch cards, many issuers also offer introductory grace periods for customers who open a new account and transfer their existing balances from a previously-held card.

Interest may be fixed or variable and compounded monthly or daily, depending on the card. Many cards will also permit **cash advances** to be withdrawn directly from ATM machines. Cash advances are often subject to origination fees and even higher interest rates. Failure to submit a minimum payment on time is called **defaulting**. Cards typically raise interest rates to a high "default rate" and may also charge extra

[15] Nilson report, April 2009.

[16] Some cards average over periods other than one month. For example, "double-cycle billing" averages balances over two months. See the Credit CARD Act of 2009 for legal details.

Figure 2. Positive and negative amortization credit card payments.



Blue circles = Alice's balance. Red squares = Bob's balance.

fees when a customer misses a payment or exceeds his or her credit limit.[17]

An internet search of contemporary[18] cards shows that rates of "prime + 13.99%" are common for a cardholder with a very good credit rating. Default rates often exceed 30% and continue to apply for months or years after the balance is repaid. To avoid defaulting, cardholders should choose cards with long grace periods and/or forgiving terms and, most importantly, pay their card bills on time.

Figure 2 shows the balance of two simulated credit card accounts. Both accounts begin with a balance of \$1,000 and an APR of 20%. In both cases, the customer sends monthly payments and makes no further charges. Using the Credit Card Payments magic formula for paying off her card in 2 years, Alice pays \$51.17 per month. Bob makes minimum payments of 1% of (balance + interest) each month. Alice is out of debt after paying about $24 \times \$51.17 = \$1,228.08$. By contrast, Bob has made minimum payments of about \$300 and still owes \$1,178.

Below is a spreadsheet formula for estimating card balances. For simplicity, each row represents one month. Because the spreadsheet does not track daily purchases, some approximations are required. We'll assume that there are 30.5 days per month, all charges are made on the first day of each month, payments are only accepted on the last

[17]The details of late payment fees and default rates are somewhat controversial. The Credit CARD Act of 2009 includes strict regulations of the definition of "late payment."

[18]As of this writing. For updated information, do your own search!

day of each month, and there is no grace period. These assumptions are deliberately pessimistic so that some margin of error is included.

Balance[m]	= Balance[m-1] + Charges[m-1] + Interest[m-1] - Payment[m-1]
Interest[m]	= (Balance[m] + Charges[m]) ∗ (1+DailyRate[m]) ∧ (30.5)
DailyRate[m]	= Daily Periodic Rate for that month (usually = APR / 365.)
Charges[m]	= Total charges during month m
Payment[m]	= Total payments made during month m

The terms and conditions are similar to those for an actual card held by the author:

(1) Average daily balance is used to calculate interest.
(2) Interest rate is variable, currently 18.99% APR compounded daily.
(3) Minimum payment each month is 1% of (Balance + Interest)
(4) If two payments are missed within 12 months, card is "in default."
Default APR is 27.99% and does not decline for 1 year.
(5) Cash advance APR is 25.99%.
(6) If card balance is paid in full each month, no interest is charged.

If the card is ever in default, the DailyRate column must be changed to the default daily periodic rate. Any fees should be added to the Charges column. This model uses daily compounding and an average daily balance method. The actual details for calculating credit card interest vary among lenders, so it is important to *read the fine print* for your card! U.S. law requires card issuers to publish a "terms and conditions" statement publishing fees, rules, and an APR.

Appendix A. Derivation of Magic Formulas

For mathematically-inclined readers, this appendix shows the calculations and assumptions used to produce the Magic Formulas from Section 0.

Fixed-Rate Compound Interest

Consider a fixed-rate loan with interest compounded n times per year, an APR of R, and principal balance A_0. Assume the borrower makes no payments.

The periodic interest rate is $r = R/n$. After one time period, the balance A_1 is principal + interest = $A_1 = A_0 + rA_0 = A_0(1 + r)$. The next period's interest is r times that: $rA_0(1 + r)$. The new balance is the old balance plus the new interest:

$$A_2 \;=\; A_1 + rA_1 \;=\; A_0(1 + r) + rA_0(1 + r) \;=\; A_0(1 + r)(1 + r) \;=\; A_0(1 + r)^2$$

The next period's interest is r times that, so the next period's balance is

$$A_3 = A_2 + rA_2 = A_2(1 + r) = A_0(1 + r)^3$$

There's a pattern here: the balance after n periods is $A_n = A_0(1+r)^n$. There are nt periods in t years, so the balance after t years is $A(t) = (1+r)^{nt}$. Since $r = R/n$, we have derived the fixed-rate compound interest formula:

$$A(t) = A_0 \left(1 + \frac{R}{n}\right)^{nt}$$

If interest is compounded rapidly, this expression approaches a limit:[19]

$$\lim_{n\to\infty} \left(1 + \frac{R}{n}\right)^{nt} = e^{Rt}$$

The formula for continuously-compounded interest is thus $A(t) = A_0(e^{Rt})$.

APR vs. APY comparison

This formula is derived in the section "APR vs. APY."

$$APY = \left(1 + \frac{R}{n}\right) - 1 \qquad R = n\left(\sqrt[n]{APY + 1}\right)$$

Monthly Payments (Approximate)

Assume that interest is continuously compounded at a rate of R per year and payments are made continuously at a rate of $12M$ per year. The rate of change of the value of the loan is (interest rate - payment rate). Write this as a differential equation:

$$\frac{d}{dt}A(t) = R \times A(t) - 12M$$

The general solution to $\frac{d}{dt}x(t) = \lambda x(t) + \mu$ can be found in any textbook on differential equations: $x(t) = Ce^{\lambda t} - \mu$, where C is some constant determined by the initial conditions. The result for continuous compounding and continuous payments is:

$$A(t) = \left(A_0 - \frac{12M}{R}\right) e^{Rt} + \frac{12M}{R}$$

As a check, find the derivative of $A(t)$ and see that it actually solves the equation:

$$\frac{d}{dt}A(t) = R\left(A_0 - \frac{12M}{R}\right) e^{Rt} = R \times A(t) - 12M$$

To find the payment rate M needed to completely pay off the loan in exactly T years, set $A(T) = 0$ and solve for M:

$$0 = \left(A_0 - \frac{12M}{R}\right) e^{RT} + \frac{12M}{R} \;=\; \frac{12M}{R}\left(1 - e^{RT}\right) + A_0 e^{RT}$$

[19] See the Appendix "Mathematics of Exponential Growth" for details.

$$\frac{12M}{R}\left(e^{RT}-1\right)=A_0e^{RT} \quad \Rightarrow \quad \frac{12M}{R}=A_0\frac{e^{RT}}{e^{RT}-1}=A_0\frac{1}{1-e^{-RT}}$$

$$M=A_0\frac{R}{12}\left(\frac{1}{1-e^{-RT}}\right)$$

To find the time needed to pay off the loan, set $A(T)=0$ and solve for T:

$$0=\left(A_0-\frac{12M}{R}\right)e^{RT}+\frac{12M}{R}\Rightarrow\frac{-12M}{R\left(A_0-\frac{12M}{R}\right)}=e^{RT}\Rightarrow RT=\ln\left(\frac{\frac{-12M}{R}}{A_0-\frac{12M}{R}}\right)$$

$$T=\frac{1}{R}\ln\left(\frac{-1}{A_0\frac{R}{12M}-1}\right)=\frac{1}{R}\ln\left(\frac{1}{1-A_0\frac{R}{12M}}\right)=\frac{-1}{R}\ln\left(1-\frac{A_0R}{12M}\right)$$

Note that if $12M=A_0R$, then $T=-\ln(0)=\infty$. This is correct: interest-only payments will never completely repay a debt! Also, if $12M<A_0R$, then the monthly payments are too small to cover monthly interest. In this case, the loan grows exponentially forever, there are no real solutions for T, and the formula returns a complex number.

Few lenders actually compound continuously, so this formula may appear to overestimate M and T. However, we also assumed that payments are made as a continuous trickle of money. Actual payments tend to be made in lumps at the end of the month. In practice, this formula will often *underestimate* M and T, so paying off a loan may require slightly more time and money than this formula predicts.

Fixed-Rate Savings Account (Approximate)

Interest rates for savings accounts are usually advertised as APY. To find the monthly rate r, read the fine print or convert APY to an APR using the APR/APY magic formula and set $r=\frac{1}{12}\text{APR}$. (This formula assumes monthly compounded interest.) The initial deposit is A_0 and an additional deposit D is made each month. As an approximation, assume all deposits are the same size. Inflation, fees, and taxes are neglected. The amount in the account each month is:

$A_1=A_0+rA_0+D=A_0(1+r)+D$
$A_2=A_1(1+r)+D=A_0(1+r)^2+D(1+r)+D$
$A_3=A_2(1+r)+D=A_0(1+r)^3+D(1+r)^2+D(1+r)+D$

$A_m=A_0(1+r)^m+D\big[(1+r)^{m-1}+\cdots+(1+r)^2+(1+r)+1\big]$

The expression in square brackets is a special case of the **geometric series**:

$$1+x+x^2+\cdots+x^{m-1}=\sum_{k=0}^{m-1}x^k=\frac{1-x^m}{1-x}$$

Substituting this result into the formula for A_m, we find:

$$\left[(1+r)^{m-1} + \cdots + (1+r)^2 + (1+r) + 1\right] = \sum_{k=0}^{m-1} (1+r)^k$$

$$A_m = A_0(1+r)^m + D\frac{1-(1+r)^m}{1-(1+r)} = A_0(1+r)^m + D\frac{(1+r)^m - 1}{r}$$

$$A_m = \left(A_0 + \frac{D}{r}\right)(1+r)^m - \frac{D}{r}$$

Since r is just the monthly rate $r = \frac{R}{12}$, the magic formula for A after t years is:

$$A(t) = \left(A_0 + \frac{12D}{R}\right)\left(1 + \frac{R}{12}\right)^{12t} - \frac{12D}{R}$$

We can think of a monthly payment M on a loan as a monthly deposit of $-D$ into a savings account. (Remember, in a savings account, the bank is the borrower.) The remaining balance on a monthly-compounded loan after t years is then:

$$A(t) = \left(A_0 - \frac{12M}{R}\right)\left(1 + \frac{R}{12}\right)^{12t} + \frac{12M}{R}$$

Notice the similarity to the continuous-interest formula: the monthly formula is just the continuous formula with $(1 + \frac{R}{12})^{12t}$ in place of e^{Rt}.

Mortgage Payments (Exact)

From the previous section, we know how to find the balance $A(t)$ of a fixed-rate mortgage after t years assuming payments of M are made each month. What minimum payment is needed to guarantee that the loan is paid off after T years? Let r be the monthly rate $R/12$. Once again, set $A(T) = 0$ and solve for M:

$$0 = A_0(1+r)^{12T} + \frac{M}{r}\left[1 - (1+r)^{12T}\right]$$

$$\frac{M}{r} = \frac{-A_0(1+r)^{12T}}{1-(1+r)^{12T}} = -A_0\frac{1}{(1+r)^{-12T} - 1} = A_0\frac{1}{1-(1+r)^{-12T}}$$

$$M = A_0\frac{R}{12}\left(\frac{1}{1-(1+\frac{R}{12})^{-12T}}\right)$$

This result also resembles the continuous version but with e^{Rt} replaced by $(1 + \frac{R}{12})^{-12T}$. We can also solve for the number of years T needed to pay off a mortgage by setting $A(T) = 0$ and solving for T:

$$(1+r)^{12T} = \frac{-\frac{M}{r}}{A_0 - \frac{M}{r}} = \frac{-1}{\frac{A_0 r}{M} - 1} = \frac{1}{1 - \frac{A_0 r}{M}}$$

$$12T = \log_{(1+r)}\left(\frac{1}{1-\frac{A_0 r}{M}}\right) = -\log_{(1+r)}\left(1 - \frac{A_0 r}{M}\right)$$

Using the logarithm change-of-base rule $\log_a x = \frac{\log x}{\log a}$, this can be rewritten as

$$12T = \frac{-1}{\log(1+r)} \log\left(1 - \frac{A_0 r}{M}\right)$$

where the log can be of any base. (The LOG button on most calculators is base-10, while LN is base-e. Either one will work fine in this formula.) Since $r = R/12$,

$$T = \frac{-1}{12 \log\left(1 + \frac{R}{12}\right)} \log\left(1 - \frac{A_0 R}{12M}\right)$$

As with continuous compounding, monthly payments must be at least $A_0 \times R/12$ or else the loan will never be paid off and the formula does not return real numbers.

Stafford Loans

The Stafford Loan magic formula is just the simple interest formula. During grace periods, Stafford interest does not compound. The monthly rate is $r = R/12$, so

$$A_m = A_0 + A_0 \frac{R}{12} + A_0 \frac{R}{12} + \cdots + A_0 \frac{R}{12} = A_0 \left(1 + \frac{R}{12} m\right)$$

This is only true during grace periods. At all other times, Stafford loans function exactly like fixed-rate mortgages with minimum monthly payments.

Credit Card Payments (Approximate)

The credit card formula is more complicated and uses some approximations:

(1) The card uses daily-compounded interest with daily rate $r = R/365$.
(2) The cardholder charges C to the card at the start of each month.
(3) The cardholder makes a payment of M at the end of each month.
(4) The card is never in grace period or default.
(5) There are 31 days in each month.

A_0 will be the balance of the card at $t = 0$, the start of our calculation. On the first day of every month, we'll charge C more to the card. On the last day of each month, we'll send a check for M to the card company. This particular choice of timing is the most expensive way to charge things without defaulting. Along with the 31-day months, this should make our formula overly pessimistic, which provides us with some margin of error.

We buy something that costs A_0 on the very first day, so the average daily balance for the first month is A_0. On the first day of each month, we immediately buy everything we'll need for that month for a total charge C. During every month (including the first one), we make a payment of M. The principal balance plus first month's interest is $A_0(1+r)^{31}$. After the initial payment M and a charge of C, the balance is:

$$A_1 = A_0(1+r)^{31} + C - M$$

At the end of the month, before payments and charges have been applied, the balance will be $A_2(1+r)^{31} = A_0[(1+r)^{31}]^2 + (C-M)(1+r)^{31}$. Applying another payment M and charging C more leaves a balance of:

$$A_2 = A_0[(1+r)^{31}]^2 + (C-M)\big[(1+r)^{31} + 1\big]$$

Continuing in this fashion produces the balance at the end of month 3:

$$A_3 = A_3(1+r)^{31} + C - M = A_0\big[(1+r)^{31}\big]^3 + (C-M)\Big(\big[(1+r)^{31}\big]^2 + (1+r)^{31} + 1\Big)$$

The general formula for the balance at the end of month m is:

$$A_m = A_0\big[(1+r)^{31}\big]^m + (C-M)\sum_{k=0}^{m-1}\big[(1+r)^{31}\big]^k$$

Let's simplify by defining a new variable $x \equiv (1+r)^{31} = (1+\frac{R}{365})^{31}$. Using the geometric series formula $1 + x + x^2 + \cdots + x^{m-1} = (1-x^m)/(1-x)$, we can write

$$A_m = A_0x^m + (C-M)\left(\frac{1-x^m}{1-x}\right)$$

To give an idea of the size of x, here it is for APRs of 10, 20, and 30%.

$$\begin{aligned} R = 0.10 &\quad\Rightarrow\quad x \approx 1.00853 \\ R = 0.20 &\quad\Rightarrow\quad x \approx 1.01713 \\ R = 0.30 &\quad\Rightarrow\quad x \approx 1.02580 \end{aligned}$$

To find the minimum monthly payment needed to pay off the credit card after m months, set $A_m = 0$ and solve for M. Doing some arithmetic, we find:

$$A_0x^m + C\left(\frac{x^m-1}{x-1}\right) = M\left(\frac{x^m-1}{x-1}\right)$$

$$M = A_0x^m\left(\frac{x-1}{x^m-1}\right) + C = A_0\left(\frac{x^{m+1}-x^m}{x^m-1}\right) + C$$

$$M = A_0\left(\frac{x-1}{1-x^{-m}}\right) + C \qquad x \equiv (1+\frac{R}{365})^{31} \approx 1 + .085(R)$$

This formula should, in principle, slightly overestimate monthly payments and the time required to pay off debt. In practice, computational rounding and the precise method of interest calculation for a particular card may reduce its precision.

Appendix B. Mathematics of Exponential Growth

If the APR on a loan is R, the interest is compounded n times per year, and the principal balance is A_0, then the amount $A(t)$ owed after t years is:

$$A(t) = A_0\Big(1 + \frac{R}{n}\Big)^{nt}$$

if no payments have been made on the loan. The binomial theorem says that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad\qquad \binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$$

where the expression in parentheses, pronounced "n choose k," is a **binomial coefficient**.[20] The compound-interest formula is then

$$A(t) = A_0\left(1 + \frac{R}{n}\right)^{nt} = A_0\left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{R^k}{n^k}\right)^t$$

When n is very large, $A(t)$ approaches an **exponential growth function**.[21]

$$\lim_{n\to\infty} A_0\left(1 + \frac{R}{n}\right)^{nt} = A_0\left(\sum_{k=0}^{\infty} \frac{1}{k!}R^k\right)^t = A_0\big(e^R\big)^t = A_0(e^{Rt})$$

This is the general formula for continuously-compounded interest with no payments. The number $e \approx 2.71828\ldots$ is called **Euler's number**. Like π, this number cannot be written down exactly as a decimal or fraction: it is **irrational**.[22]

Notice that $A(t)$ does *not* become infinite as $n \to \infty$, so continuously compounded interest does not result in unlimited profits to lenders. Instead, $A(t)$ approaches a continuous function. In the limit of very rapid compounding, the interest formula becomes:

$$\tfrac{d}{dt}A(t) = R \cdot A(t)$$

which mathematicians call a "first-order linear homogeneous differential equation." The solution is the same as Euler's compound-interest formula: $A(t) = A_0(e^{Rt})$, where A_0 is whatever value A had at time $t = 0$.

If R is positive, then e^{Rt} is called an exponential growth function.[23] In general, exponential growth describes anything with a growth rate proportional to its current size. Famous examples include bacterial reproduction, disease propagation, and population size with a surplus of resources. Exponential growth curves are shown in Figure 3.

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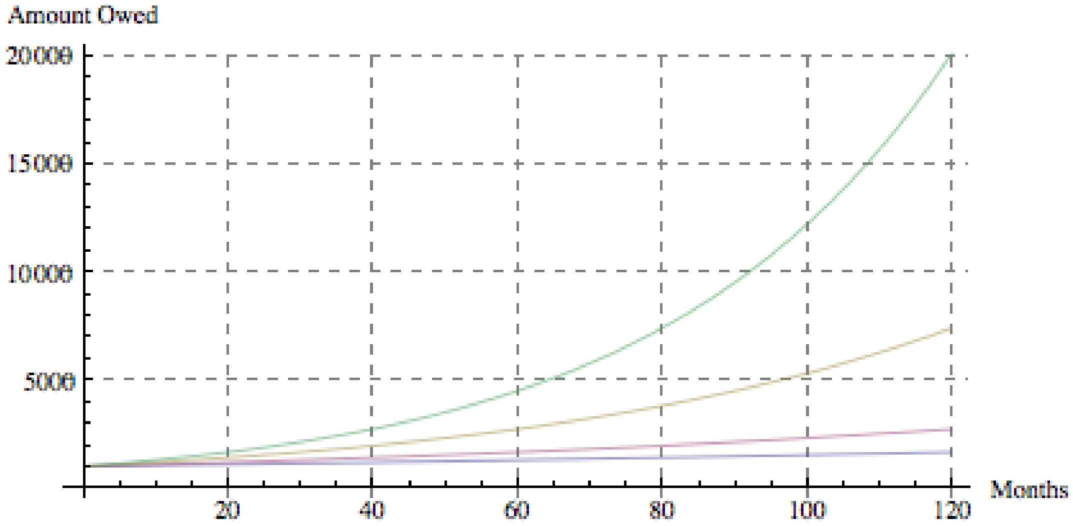
[20](n choose k) is the number of k-card poker hands that can be made from a deck of n cards or, equivalently, the number of k-color combinations that can be chosen from a box of n crayons.

[21]Versions of this result were proved by Euler, Leibniz, Newton, and Johann Bernoulli.

[22]Both π and e are also **transcendental**: neither is a root of any polynomial with rational coefficients. Jacob Bernoulli discovered e in 1683 while studying compound interest.

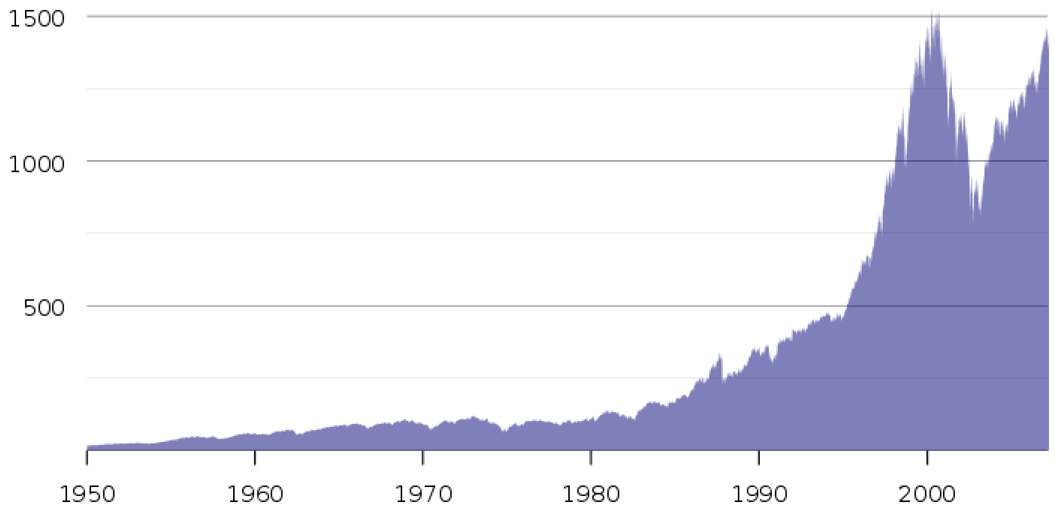
[23]If R is negative, the solution is still $A_0(e^{Rt})$, but it is now called **exponential decay**.

Figure 3. Exponential growth of a loan balance.



Top to bottom: $1,000 loan at 30%, 20%, 10%, and 5% APR

Figure 4. Standard and Poor's "S&P 500" index from 1949-2007.



All of these graphs curve upward, and the curvature becomes more apparent at high rates and over long time periods.[24] With compound interest, unpaid debts grow faster and faster without limit. The use of compound interest essentially reflects a belief that *the time value of money is approximately exponential.*

This is a reasonable assumption provided that large-scale economic growth in general is also approximately exponential. A major weighted-average index of the U.S. stock

[24]Note that after 10 years, the 30% APR loan balance is about 2,000% of its principal!

market (Figure 4) does indeed resemble an exponential growth over the span of several decades, but the agreement is far from exact, especially over shorter time periods. A simple exponential growth function does not account for the bubbles, crashes, and other erratic behaviors displayed by real-world economies.

Appendix C. Risk and the Kelly Criterion

For the examples in which Bob loans Alice money to start a bakery, it may appear that Bob is profiting from Alice while Alice is doing all the work of baking and managing a business. This is true! But it is also true that Bob stands to lose the most under their arrangement. If the bakery fails quickly and Alice defaults on the loan, Bob loses more money than Alice does. So who gets the better deal?

To answer this question sensibly, we can say that *Alice is paying Bob to offset her risk.* In an idealized model economy, there is some **risk-free interest rate** that accurately models the opportunity costs of lending money to anyone. In the real world, if the value of a low-risk investment grows exponentially at a rate ρ, then Bob could invest his money there for a profit of $A_0(e^{\rho t} - 1)$ after t years.[25] Instead, Bob invests in Alice's bakery, knowing it could fail. In return, he asks her to pay a rate higher than ρ.

A loan can benefit both parties if Bob's **risk tolerance** is greater than Alice's. If Bob has a large surplus of wealth and Alice does not, then the failure of Alice's bakery may be an acceptable risk for him and an unacceptable risk for her. Because Alice does not want to invest all her money in something that might fail, we say she is **risk-averse**. Even if Alice and Bob agree that her bakery will probably turn a profit, she requires some protection against the consequences of failure.

Imagine a simple investment I that either succeeds for a profit of I_{win} with probability p or fails for a loss of I_{lose} with probability q. Since one or the other must happen, $p + q = 1$. The investment's **expectation value** $\langle I \rangle$ is:

$$\langle I \rangle = pI_{win} - qI_{lose} = pI_{win} - (1 - p)I_{lose}$$

For example, Dave throws a coin. If it comes up heads, Charlie wins \$1 from Dave. If it is tails, Charlie pays Dave \$2. For a fair coin, the expectation value of Charlie's investment is $(0.50)1 - (.50)2 = -0.50$. Charlie declines the bet because, for him, the expectation value is negative. Roughly speaking, expectation value is the amount one expects to win per investment if the investment is repeated a very large number of times.[26]

As another example, consider betting on 7 at roulette. On a Vegas wheel, there are 38 numbers (1-36, 0, and 00) and each is equally likely to be chosen on a particular spin.

[25]Risk-free investments do not exist. However, the risk that the U.S. government will default on its bonds is presumably small. Because markets decide bond rates, they can be used as a consensus estimate of ρ. Similar reasoning applies to low-risk interbank prime rates.

[26]For a precise definition, look up the "Law of Large Numbers" in a statistics textbook.

If 7 is the result, a gambler is paid 35:1, meaning \$35 for each \$1 bet. The expectation value is a loss of 5.26 cents for each \$1 invested:

$$\frac{1}{38}35 - \frac{37}{38}1 = -\frac{2}{38} \approx -5.26\%$$

Betting on red pays "even money," or 1:1, which turns out to be no better:

$$\frac{16}{38}1 - \frac{18}{38}1 = -\frac{2}{38} \approx -5.26\%$$

For a casino, however, roulette can be quite profitable! If 10,000 bets are placed on red for \$10 each, the expected profit for the casino is $10 \times 10,000 \times \frac{2}{38} \approx \$5,263$.

Real-world investments need not simply succeed or fail. There are often possible outcomes that could be classified as partial successes, such as closing with no profit after recouping some fraction of the startup costs. In general, expectation value is found by multiplying the probability of every outcome by the profit or loss caused by that outcome.

For example, Dave will roll a six-sided die and pay Charlie \$2 times the number that comes up unless the number is 6, in which cases Dave wins \$28. Should Charlie play?

$$\langle I \rangle = \frac{1}{6}2 + \frac{1}{6}4 + \frac{1}{6}6 + \frac{1}{6}8 + \frac{1}{6}10 - \frac{1}{6}28 = \frac{1}{3} \approx \$0.33$$

The expectation value is positive for Charlie. Now imagine that Dave offers the same game except he pays Charlie \$1 million times the number on the die if it is 1,2,3,or 5 but Charlie pays Dave \$14 million if it is 6. Charlie's expectation value for this game is positive (almost \$167,000), so of course Charlie "should" play.

Would you? If you lose, you owe Dave \$14 million. Even if you succeed several times at first, one loss could wipe out your profits. If you can't afford to lose \$14 million, this game is essentially Russian roulette. At such high stakes, most people are (wisely!) risk-averse. But if the bet was much smaller, Charlie should clearly accept.

The **Kelly criterion**[27] is a means for choosing the optimum amount to bet if p is the probability of success and each success pays $\$P$ for every \$1 invested:

$$K(p, P) = \frac{1}{P}\Big[p(P+1) - 1\Big]$$

Kelly's rules are: Decide the maximum amount B an investor is willing to lose (called a **bankroll**), then bet the amount $b = K \times B$. After each success or failure, add/subtract the profit/loss to B and adjust the new bet size accordingly.

[27] From a 1956 paper by J. L. Kelly, Jr. It is now known that Daniel Bernoulli discovered a similar result while solving the "St. Petersburg Paradox," a problem posed by his cousin Nicolas. (In case you were wondering, the many Bernoullis mentioned here are indeed related. Daniel's father was Johann, and Jacob was uncle to both Daniel and Nicolas.)

This definition is for a simple win-or-lose proposition like a coin toss.[28] If there are multiple outcomes, define the **expected utility** U of the investment like so:[29]

$$\langle U \rangle \equiv \langle \log(\beta) \rangle = \sum_{\text{all } n} p_n \log(\beta_n)$$

where β_n is the fraction of the investor's bankroll remaining if the nth possible result occurs, and p_n is the probability of that result. The Kelly criterion tells us to **bet whatever amount maximizes $\langle U \rangle$ if it is positive and refuse to bet if $\langle U \rangle$ is negative.**

If Charlie's bankroll is \$200, should he accept Dave's \$2 dice game? Find $\langle U \rangle$:

$$\frac{1}{6}\left(\log\left[\frac{202}{200}\right] + \log\left[\frac{204}{200}\right] + \log\left[\frac{206}{200}\right] + \log\left[\frac{208}{200}\right] + \log\left[\frac{210}{200}\right] + \log\left[\frac{172}{200}\right]\right)$$
$$\approx -0.000583$$

This value is negative, so Kelly's criterion says not to bet. But if Dave lowers the stakes to \$1 times the number for a win and -\$14 for a loss, then $\langle U \rangle$ becomes a (small) positive number and Kelly's advice is to accept. For comparison, the Kelly criterion requires a bankroll of over \$133 million to play the million-dollar version of Dave's game.

If an investor has bankroll B, how much money b should he or she bet on each trial of a win-or-lose investment that pays P:1 and succeeds with probability p? Kelly recommends maximizing $\langle U \rangle$, which we do by setting the b derivative of $\langle U \rangle$ equal to 0. After a success or failure, the fraction β of bankroll remaining will be:

$$\beta_{win} = \frac{B + Pb}{B} \qquad \beta_{lose} = \frac{B - b}{B}$$

For an investment with only two possible outcomes, $\langle U \rangle$ is given by

$$\langle U \rangle = \langle \log(\beta) \rangle = p \log(\beta_{win}) + (1-p)\log(\beta_{lose})$$

Now we set $\frac{\partial}{\partial b}\langle U \rangle$ equal to zero and use the "chain rule" from calculus:

$$\langle U \rangle = \langle \log(\beta) \rangle = p \log(\beta_{win}) + (1-p)\log(\beta_{lose})$$
$$\frac{\partial}{\partial b}\langle U \rangle = \frac{pP}{B(1 + P\frac{b}{B})} - \frac{1-p}{B(1 - \frac{b}{B})} = 0$$
$$\frac{Pp + p - 1 - P\frac{b}{B}}{B\left(1 + P\frac{b}{B}\right)\left(1 - \frac{b}{B}\right)} = \frac{p(P+1) - 1 - P\frac{b}{B}}{B\left(1 + P\frac{b}{B}\right)\left(1 - \frac{b}{B}\right)} = 0$$

In order for this to be true, the numerator must itself be zero:

$$p(P+1) - 1 = P\frac{b}{B} \quad \Leftrightarrow \quad \frac{b}{B} = \frac{1}{P}\left[p(P+1) - 1\right] \quad \Leftrightarrow \quad b = B \times K(p, P)$$

which is the Kelly strategy for a win-or-lose investment. For a more complicated investment, the formula is derived the same way: set $\frac{\partial}{\partial b}\langle \log(\beta) \rangle = 0$.

[28] A random process with exactly two possible outcomes is called a (Jacob) **Bernoulli trial**.

[29] There are many possible definitions of "utility." Choosing $U = \log(\beta)$ and finding its expectation value is the choice equivalent to the Kelly criterion. Here we use log to mean the base-10 logarithm, but any other choice of base leads to the same conclusions.

Note that the Kelly criterion recommends *never* betting all of one's bankroll on any one investment. If the probability of losing one's entire bankroll is ϵ, then $\langle \log \beta \rangle$ contains a term $\epsilon[\log(0)] = -\infty$. Since there is no possibility of a $+\infty$ term, we can safely conclude that $\langle U \rangle = -\infty$ even if ϵ is extremely small.

The probability of losing one's entire bankroll is called **risk of ruin**. Kelly's original motivation was to calculate the least risk-averse bet size that still has zero risk of ruin. Consequently the Kelly criterion rejects "all-in" bets in which β could become zero. In practice, Kelly-betting is often treated as an upper limit. Many investors will bet up to the amount Kelly recommends, but never more.

Practical difficulties motivate this decision. Kelly's derivation assumes that a bankroll is infinitely divisible: if we continue losing through a run of bad luck, we continue making smaller and smaller bets until we win our money back. In reality, the loss of a substantial portion of bankroll could have serious side effects which we have neglected mathematically. (Imagine a mutual fund that loses 80% of its value in one year. Might this change the odds that clients continue to invest in the fund?)

Another major problem is that, for real investments, the precise probability of an outcome is rarely known exactly. What is the probability that Alice's bakery fails within one year? Even very unlikely events can reduce Kelly's recommended bet to zero if those events could lead to ruin or near ruin.

The reasoning behind the Kelly criterion provide a mathematical explanation for why insurance contracts exist. Small investors often have to turn down profitable choices, such as starting a business, to avoid risk of ruin due to unpredictable events. Suppose Alice decides not to open a bakery because losing it in a fire or flood would wreck her bankroll and leave her in debt. If the cost of insurance is not too high, she can transfer that risk to another company and safely invest her time and money in her business.

In summary, even if the mathematical details have left the reader confused, a few rules might well summarize the most important results of Kelly-style risk analysis:

(1) Never bet more than you can afford to lose.
(2) Bigger bankrolls can afford bigger bets.
(3) It is most profitable to be risk-averse, but not too much so.
(4) Neglect unlikely-but-expensive failures at your peril.

As of this writing, international banks have lost an estimated $1 trillion dollars in the last two years due to failed loans, government bonds from a Eurozone nation have been rated as "junk" by major rating agencies, and the remains of an exploded $350 million oil rig are leaking 1-4 million quarts of oil into the Gulf of Mexico per day. Consequently, Rule #4 should be easy to remember.